Math Logic: Model Theory & Computability
Lecture 10

Thus, it is desirable to come up with a hopefully equivalent theory to $\mathrm{Th}(\underline{N})$ whose axioms would easily recognizable (say, by a computer program). Such an attempt was made by Peano, who suggested the following theory, now called Peano Arithmetic (PA), in the signature $\sigma_{arth} := (0, S, +, \cdot)$:

(PA1) $\forall x\,(0 \neq S(x))$ [0 is not in the image of S]

(PA2) $\forall x \forall y\,(S(x) = S(y) \to x = y)$ [S is injective]

(PA3) $\forall x\,(x + 0 = x)$ [0 is the additive identity]

(PA4) $\forall x \forall y\,[x + S(y) = S(x+y))$ [def. of + via S]

(PA5) $\forall x\,(x \cdot 0 = 0)$ [0 is the multiplicative annihilator]

(PA6) $\forall x \forall y\,(x \cdot S(y) = x \cdot y + x)$ [def. of $\cdot$ via +]

($\mathrm{PA}7^{\infty}$) The axiom schema of induction: for each extended σ_{arth}-formula $\varphi(x, \vec{y})$ the following is an axiom of PA:

$$\forall \vec{y}\left[\left[\varphi(0,\vec{y}) \wedge \forall x\left(\varphi(x,\vec{y}) \to \varphi(S(x),\vec{y})\right)\right] \to \forall x\, \varphi(x,\vec{y})\right],$$

where $\forall \vec{y}$ abbreviates $\forall y_1 \forall y_2 \dots \forall y_k$, where $\vec{y} = (y_1, \dots, y_k)$.

Peano hoped that PA would be an equivalent theory to $\mathrm{Th}(\underline{N})$, but Gödel proved that this is not the case, in fact, there is no computer recognizable theory equivalent to $\mathrm{Th}(\underline{N})$ — this is known as the Gödel incompleteness theorem.

Semantic consistency, implication, and completeness.

Def. A σ-theory is called satisfiable (semantically consistent) if it has a

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nonempty model.

All examples of theories given above are satisfiable.

Def. For a σ-theory T and a σ-sentence φ, we say that T models/satisfies/semantically implies φ, denoted $T \models \varphi$, if every model of T satisfies φ. In other words, $T \models \varphi$ if and only if $\varphi \in \bigcap_{\underline{M} \models T} \mathrm{Th}(\underline{M})$.

Obs. For a σ-theory T, the following are equivalent:

(1) T is not satisfiable.
(2) $T \models \varphi$ for each σ-sentence φ.
(3) $T \models \bot$, where $\bot := \exists x (x \neq x)$.

Proof. (1) $\Rightarrow$ (2). Since T doesn't have any models, it is true that every model of T satisfies whatever we want.

(2) $\Rightarrow$ (3). Special case.

(3) $\Rightarrow$ (1). No structure satisfies $\bot$, hence $T \models \bot$ implies that T has no models. $\square$

Examples. (a) $\mathrm{GROUPS} \models \overbrace{\forall x \forall y \forall z \left((y \cdot x = 1 \wedge x \cdot z = 1) \to y = z \right)}^{\varphi}$.

Proof. Let $\underline{G} := (G, 1, \cdot, ()^{-1})$ be a model of GROUPS, so a group. Fix arbitrary elements $g, h, k \in G$ (i.e. take $x := g$, $y := h$, $z := k$) and suppose $h \cdot g = 1^{\underline{G}}$ and $g \cdot k = 1^{\underline{G}}$. Then $h = h \cdot 1^{G} = h \cdot (g \cdot k) = (h \cdot g) \cdot k = 1^{\underline{G}} \cdot k = k$. Thus, $\underline{G} \models \varphi$. $\square$

(b) For each prime p and $n \in \mathbb{N}$, $\mathrm{FIELDS}_p \models \underbrace{1 + 1 + \ldots + 1}_{n} = 0$ if and only if p divides n.

To prove this, again fix any field of characteristic p and show that the statement holds in it.

(c) $\text{FIELDS}_0 \models \underbrace{1+1+\ldots+1}_{n} \neq 0$ for all $n \in \mathbb{N}^+$.

To prove this fix an arbitrary field of char. 0 and show this by induction.

Def. σ-structures $\underline{A}$ and $\underline{B}$ are called *elementarily equivalent* if they have the same theory, i.e. $\mathrm{Th}(\underline{A}) = \mathrm{Th}(\underline{B})$. We denote this by $\underline{A} \equiv \underline{B}$.

We have proven earlier that if $\underline{A}$ and $\underline{B}$ are isomorphic, then they are elementarily equivalent. However the converse isn't true in general. For example, one can show (HW) that $(\mathbb{Q}, <) \equiv (\mathbb{R}, <)$ but they can't be isomorphic because $\mathbb{Q}$ and $\mathbb{R}$ are not equinumerous.

Def. Let T be a σ-theory. We say that T is *semantically σ-complete* if for each σ-sentence φ, we have that $T \models \varphi$ or $T \models \neg\varphi$.

Note. If T is not satisfiable, then T is automatically complete because both $T \models \varphi$ and $T \models \neg\varphi$ for each σ-sentence φ.
Thus, this notion is only useful when T is satisfiable, in which case the "or" is exclusive, i.e. only one of $T \models \varphi$ and $T \models \neg\varphi$ holds.

Prop. A σ-theory T is semantically σ-complete if and only if $\underline{A} \equiv \underline{B}$ for all models $\underline{A}, \underline{B}$ of T.

Examples. (a) GROUPS is not semantically complete because, for example, there are abelian and nonabelian groups.

(b) For p prime or 0, $FIELDS_p$ is not semantically complete because there are fields of char. p that in which $x^2+1=0$ has a root and there are those in which there is no root.

(c) It is Tarski's theorem that for each p prime or 0, ACF_p is semantically complete. We will prove this later.

(d) Gödel's incompleteness theorem states that PA is not semantically complete.

(e) $Th(\underline{A})$ is semantically σ-complete for each σ-structure $\underline{A}$. In particular, $Th(\underline{N})$ is semantically σ_{arith}-complete.

<u>Def.</u> A σ-theory T is σ-maximal satisfiable if it is satisfiable and for each σ-sentence φ, we have that $\varphi \in T$ or $\neg\varphi \in T$.

<u>Example.</u> For a σ-structure $\underline{A}$, $Th(\underline{A})$ is σ-maximal satisfiable.

<u>Obs.</u> Every satisfiable σ-theory T admits a σ-maximal satisfiable extension $\tilde{T} \supseteq T$. In particular, every satisfiable σ-theory T admits a satisfiable σ-completion.

Proof. T has a model $\underline{M}$ so let $\tilde{T} := Th(\underline{M})$. $\square$